

DARBOUX-WEINSTEIN THEOREM FOR LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

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ABSTRACT. A locally conformally symplectic (LCS) form is an almost symplectic form ω such that a closed one-form θ exists with $d\omega = \theta \wedge \omega$. We present a version of the well-known result of Darboux and Weinstein in the LCS setting and give an application concerning Lagrangian submanifolds.

Keywords: Locally conformally symplectic, Darboux-Weinstein theorem, lagrangian submanifold.

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1. INTRODUCTION

The aim of this note is to extend in a natural way the classical Darboux-Weinstein theorem, which we now recall (see e.g. [MS]):

Theorem 1.1: (Darboux-Weinstein) *Let M be a manifold and ω_0, ω_1 closed 2-forms on M . Let $Q \subset M$ be a compact submanifold such that ω_0 and ω_1 are nondegenerate and equal on $T_q M$ for all $q \in Q$.*

Then there exist $\mathcal{N}_0, \mathcal{N}_1$ neighborhoods of Q and $\varphi : \mathcal{N}_0 \rightarrow \mathcal{N}_1$ a diffeomorphism such that

$$\varphi^* \omega_1 = \omega_0 \quad \text{and} \quad \varphi|_Q = id.$$

We are interested in the more general context of locally conformally symplectic (briefly LCS) manifolds, a particular case of almost symplectic manifolds:

Definition 1.2: A manifold M with a non-degenerate two-form ω is called LCS if there exists a closed one-form θ such that $d\omega = \theta \wedge \omega$.

The notion first appears as such in [Li], it was later studied by J. Lefebvre [Lef] and especially I. Vaisman [Va]. One can easily see that the name is justified, as the definition above is equivalent to the existence of an open cover $(U)_\alpha$ and a family of smooth functions f_α on each U_α such that $d(e^{-f_\alpha}\omega) = 0$ (see [Lee]).

Our main result reads:

Theorem 1.3: *Let M be a manifold, θ_0 and θ_1 closed 1-forms and ω_0, ω_1 2-forms on M such that $d_{\theta_i}\omega_i = 0$. Let $Q \subset M$ be a compact submanifold such that ω_0 and ω_1 are nondegenerate and equal on T_qM for all $q \in Q$, and $\theta_0|_{TQ} = \theta_1|_{TQ}$.*

Then there exist $\mathcal{N}_0, \mathcal{N}_1$ neighborhoods of Q and $\varphi : \mathcal{N}_0 \rightarrow \mathcal{N}_1$ a diffeomorphism such that

$$\varphi^*\omega_1 \sim \omega_0 \quad \text{and} \quad \varphi|_Q = id.$$

where by “ \sim ” we mean conformally equivalent.

We shall end with an application concerning the behavior of any LCS form near a Lagrangian submanifold, thus extending a theorem due to Weinstein [We].

2. PROOF OF THE MAIN THEOREM

We shall heavily rely in our own proof on the intricacies of the original Darboux-Weinstein argument, as presented in [MS, Lemma 3.14, pages 93-95]. One of the instruments of both proofs is the so-called Moser Trick, which we therefore explain briefly¹:

Theorem 2.1: *Let M be a compact manifold and $(\omega_t)_{0 \leq t \leq 1}$ a smooth family of symplectic forms on M satisfying*

$$\frac{d}{dt}\omega_t = d\sigma_t$$

for σ_t varying smoothly.

Then there is an isotopy φ_t such that $\varphi_t^\omega_t = \omega_0$ with $\varphi_0 = id$.*

Proof. Choose the vector fields Y_t uniquely satisfying

$$(2.1) \quad i_{Y_t}\omega_t = -\sigma_t$$

¹The Moser Trick was extended to LCS geometry, [BK], but our proof uses the original symplectic version.

and its integral curves φ_t (i.e. $\frac{d}{dt}\varphi_t = Y_t \circ \varphi_t$ and $\varphi_0 = id$), defined overall. What we get is

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{Y_t}\omega_t\right) = \varphi_t^*(d\sigma_t + di_{Y_t}\omega_t) = 0,$$

hence $\varphi_t^*\omega_t = \omega_0$. ■

Proof of Theorem 1.3. We begin by fixing a Riemannian metric on M , which we shall use to construct a tubular neighborhood of Q in M , together with a family of diffeomorphisms representing a deformation retract onto Q .

Take

$$U_\varepsilon = \{(q, v) \in Q \times TM \mid v \in (T_q Q)^\perp \text{ and } \|v\| < \varepsilon\},$$

where the norm is given by the fixed Riemannian metric. Since Q is compact, for a sufficiently small ε , the exponential is a diffeomorphism from U_ε to a neighborhood of Q which we denote \mathcal{M}_0 . We may define

$$\varphi_t : \mathcal{M}_0 \rightarrow \mathcal{M}_0, \quad \varphi_t(\exp(q, v)) = \exp(q, tv), \quad 0 \leq t \leq 1,$$

which are diffeomorphisms onto their image, except for φ_0 , which collapses the tubular neighborhood onto Q . With that in mind, the vector fields

$$X_t = \frac{d}{dt}\varphi_t \circ \varphi_t^{-1}$$

are correctly defined for $0 < t \leq 1$ and their integral curves are φ_t .

Let $q \in Q$. We may find $V \supset \bar{U} \supset U \ni q$ small enough that ω_0 and ω_1 are nondegenerate, $\theta_0 = df_0$ and $\theta_1 = df_1$ on V ; we choose f_0 and f_1 such that $f_0(q) = f_1(q)$. By our assumptions,

$$df_0|_{TQ} = \theta_0|_{TQ} = \theta_1|_{TQ} = df_1|_{TQ},$$

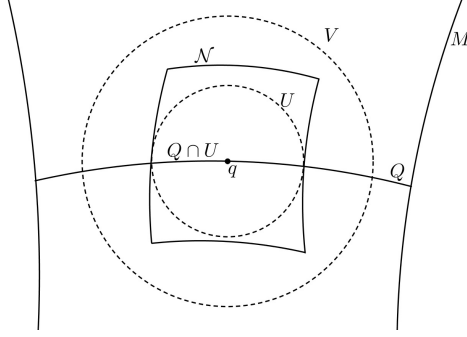
so $f_0 = f_1$ on $Q \cap V$. Consider the symplectic forms conformal to ω_0 and ω_1 on V :

$$\begin{aligned} \eta_0 &= e^{-f_0}\omega_0 \\ \eta_1 &= e^{-f_1}\omega_1. \end{aligned}$$

We can see from the above that η_0 and η_1 agree on $T_q M$ for every $q \in Q \cap V$. Let

$$W_\delta = \{(q, v) \in (Q \cap U) \times TM \mid v \in (T_q Q)^\perp \text{ and } \|v\| < \delta\};$$

for δ sufficiently small (and smaller than the ε determined previously for the entire Q), \exp is a diffeomorphism from W_δ to its image, which we denote \mathcal{N} (note that this is a neighborhood of $Q \cap U$, though not a tubular one, $\mathcal{N} \subset \mathcal{M}_0$ and $\varphi_t(\mathcal{N}) \subset \mathcal{N}$). We may also assume, picking a smaller δ if necessary, that $\mathcal{N} \subset V$ (see the figure below).



Denoting by $\tau := \eta_1 - \eta_0$, we have $\varphi_0^* \tau = 0$ and obviously $\varphi_1^* \tau = \tau$. Therefore

$$\tau = \int_0^1 \frac{d}{dt} \varphi_t^*(\tau) dt = d \int_0^1 \varphi_t^*(i_{X_t} \tau) dt;$$

let $\rho_t := \varphi_t^*(i_{X_t} \tau)$. Explicitly,

$$(\rho_t)_p(v) = \tau_{\varphi_t(p)}(X_t(\varphi_t(p)), \varphi_{t*}(v)) = \tau_{\varphi_t(p)}\left(\frac{d}{dt} \varphi_t(p), \varphi_{t*}(v)\right),$$

which is correctly defined in $t = 0$. Observe that for $p = q \in Q$, since $\varphi_t(q) = q$, we have $(\rho_t)_q = 0$. Taking

$$\sigma = \int_0^1 \rho_t dt,$$

we have obtained a one-form σ on \mathcal{N} , null on $Q \cap U$, such that $\eta_1 - \eta_0 = d\sigma$.

We now turn to the Moser Trick (Theorem 2.1) for the segment of forms $\eta_t = \eta_0 + t(\eta_1 - \eta_0)$, noticing that $\frac{d}{dt} \eta_t = d\sigma$. We may shrink the neighborhood and assume that ω_t are non-degenerate and that the integral curves obtained are defined on $[0, 1]$. We thus get $\varphi : U_q \rightarrow U'_q$ (neighborhoods of $Q \cap U$) with $\varphi^* \eta_1 = \eta_0$ and $\varphi|_{Q \cap U} = id$.

We conclude that

$$\omega_0 = e^{f_0} \eta_0 = e^{f_0} \varphi^* \eta_1 = e^{f_0 - f_1 \circ \varphi} \varphi^* \omega_1$$

on the neighborhood U_q of $Q \cap U$.

We have obtained the result we wanted locally on Q , by applying (essentially) the Darboux-Weinstein technique on patches of Q . Of course, we want the local diffeomorphisms that we have constructed, as well as the conformal factors, to agree on the intersections. This does not usually happen; however, in our case, having the benefit of having used a global instrument (namely, the metric on M), we will only need a brief overview of the facts to reach this conclusion.

We can construct a cover U_α of Q in M such that:

- (1) $\theta_0 = df_0^\alpha$ and $\theta_1 = df_1^\alpha$ on U_α ;
- (2) $f_0^\alpha = f_1^\alpha$ on $Q \cap U_\alpha$;
- (3) We have the symplectic forms $\eta_0^\alpha = e^{-f_0^\alpha} \omega_0$ and $\eta_1^\alpha = e^{-f_1^\alpha} \omega_1$ on U_α ;

- (4) There is a 1-form σ^α on U_α with $d\sigma^\alpha = \eta_1^\alpha - \eta_0^\alpha$. More precisely,

$$\sigma^\alpha = \int_0^1 \varphi_t^* i_{X_t} (\eta_1^\alpha - \eta_0^\alpha) dt;$$

- (5) The vector field Y_t^α on U_α is uniquely determined by the Moser Formula (2.1):

$$(2.2) \quad i_{Y_t^\alpha} \eta_t^\alpha = -\sigma^\alpha;$$

where $\eta_t^\alpha = \eta_0^\alpha + t(\eta_1^\alpha - \eta_0^\alpha)$.

- (6) Lastly, we have a diffeomorphism $\varphi^\alpha : U_\alpha \rightarrow U'_\alpha$ such that

$$\omega_0 = e^{f_0^\alpha - f_1^\alpha \circ \varphi^\alpha} (\varphi^\alpha)^* \omega_1 \text{ on } U_\alpha$$

$$\varphi|_{U_\alpha \cap Q} = id,$$

φ^α being the integral curve (at time $t = 1$) of Y_t^α .

Note that φ_t and X_t , being a byproduct of the chosen metric, **are independent of α** , varying only in domain in the above expressions.

On $U_\alpha \cap U_\beta$, we have the following: firstly, since $df_0^\alpha = \theta_0 = df_0^\beta$,

$$f_0^\alpha = c_{\alpha\beta} + f_0^\beta.$$

The same is true of the f_1 -s:

$$f_1^\alpha = c'_{\alpha\beta} + f_1^\beta.$$

However, since $f_0^\alpha = f_1^\alpha$ on $Q \cap U_\alpha$ and $f_0^\beta = f_1^\beta$ on $Q \cap U_\beta$, we conclude that $c_{\alpha\beta} = c'_{\alpha\beta}$. We then immediately get

$$\begin{aligned} \eta_0^\alpha &= e^{-c_{\alpha\beta}} \eta_0^\beta \\ \eta_1^\alpha &= e^{-c_{\alpha\beta}} \eta_1^\beta, \end{aligned}$$

so

$$\eta_t^\alpha = e^{-c_{\alpha\beta}} \eta_t^\beta \text{ and } \sigma^\alpha = e^{-c_{\alpha\beta}} \sigma^\beta.$$

We now see clearly from (2.2) that, on $U_\alpha \cap U_\beta$, the vector fields Y_t^α and Y_t^β satisfy the same formula, and must be equal. Then $\varphi^\alpha = \varphi^\beta$ on $U_\alpha \cap U_\beta$, and we can glue them to a global diffeomorphism

$$\varphi : \mathcal{N}_0 := \bigcup_\alpha U_\alpha \rightarrow \mathcal{N}_1 := \bigcup_\alpha U'_\alpha$$

with $\varphi|_Q = id$ and

$$\omega_0 = e^{f_0^\alpha - f_1^\alpha \circ \varphi} \varphi^* \omega_1 \text{ on } U_\alpha, \forall \alpha.$$

However, it is clear now that the conformal factors are also equal on the intersections:

$$f_0^\alpha - f_1^\alpha \circ \varphi = c_{\alpha\beta} + f_0^\beta - (c_{\alpha\beta} + f_1^\beta) \circ \varphi = f_0^\beta - f_1^\beta \circ \varphi,$$

and we have reached our conclusion. ■

Remark 2.2: The condition of equality on $T_q M$ of the two LCS forms might seem a bit restrictive. Nevertheless, there are a few cases where it may be lessened to equality on TQ , for instance if Q is a point (where the conclusion is an easy consequence of the classical Darboux theorem) or if Q is Lagrangian for both ω_0 and ω_1 . In the latter case, the proof in [CdS, Theorem 8.4, pages 48-49] can be readily adapted to the LCS case, thus reducing the problem to Theorem 1.3.

3. AN APPLICATION

In the symplectic case, the last remark has as consequence the following theorem, describing any symplectic form around a Lagrangian submanifold in terms of the standard symplectic form on its cotangent bundle. We state the precise result below, due to Weinstein [We, Theorem 6.1, pages 338-339]:

Theorem 3.1: *Let (M, ω) be a symplectic manifold and $Q \subset M$ a compact Lagrangian submanifold.*

*Then there exists a neighborhood \mathcal{M} of Q , a neighborhood \mathcal{N} of the zero section in T^*Q and a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ such that*

$$\varphi^* \omega_0 = \omega,$$

*where ω_0 is the standard symplectic form on T^*Q .*

There is an analogue of this result in the LCS case, which uses the LCS structures of the cotangent bundle introduced by S. Haller and T. Rybicki in [HaR]: take θ a closed one-form on Q and η the Liouville form on T^*Q . Then it can be proven that

$$\omega_\theta = d\eta - \pi^* \theta \wedge \eta$$

is LCS with the Lee form $\pi^* \theta$. It can also be easily seen that the zero section is then Lagrangian. Note that ω is globally conformally symplectic if and only if θ is exact.

We can now state our extension of the previous theorem to LCS manifolds:

Theorem 3.2: *Let (M, ω) be an LCS manifold with Lee form θ and $Q \subset M$ a compact Lagrangian submanifold.*

*Then there exists a neighborhood \mathcal{M} of Q , a neighborhood \mathcal{N} of the zero section in T^*Q and a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ such that*

$$\varphi^* \omega_\theta = \omega,$$

where ω_θ is the LCS form described above.

Proof. We first wish to transport the form ω_θ from T^*Q to a neighborhood of Q in M . Fix a Riemannian metric on M ; we then have a canonical isomorphism of vector bundles between $(TQ)^\perp$ and T^*Q , given by:

$$(TQ)^\perp \ni (q, v) \mapsto (q, w^*)$$

where w is uniquely found by $g(v, \cdot) = \omega(w, \cdot)$ and $w^* = g(w, \cdot)$ (the key point in this identification is the fact that $T_q Q$ is Lagrangian in $T_q M$ for each $q \in Q$).

Furthermore, by means of the exponential map, a neighborhood of Q in M is diffeomorphic to a neighborhood of the zero section in $(TQ)^\perp$. Consequently, we can transport the form ω_θ to a neighborhood \mathcal{U} of Q .

We need only remember that Q is also Lagrangian for this new form and apply theorem Theorem 1.3 in light of Remark 2.2 to complete the proof. ■

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